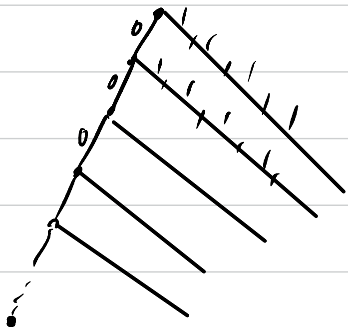
Metric Spaces and Topology
Lecture 13

Cantor–Bendixson theorem. Polish spaces have the PSP. In particular, Polish spaces satisfy the continuum hypothesis.
In fact, X uniquely decomposes into a disjoint union $P \sqcup U$ of a perfect closed space P (so if $P \neq \emptyset$, $2^{\mathbb{N}} \hookrightarrow P$) and a ctbl open U.

Thoughts. For example:

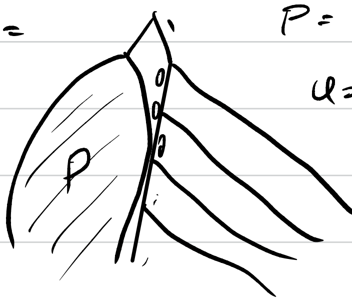
$X =$



$\cong \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}^{2}$

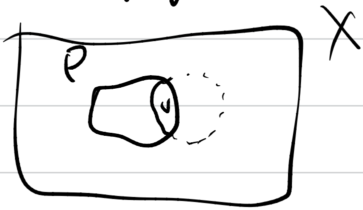
Here $P = \emptyset$
$U = X$.

Another example: $X =$



$P = [0]$,
$U = [1] \cap X = X \setminus P$.

Proof. We remove all ctbl open sets from X. To see that their union is still ctbl, recall that X is 2^{nd} ctbl, Fix a ctbl basis $\{U_n\}_{n\in\mathbb{N}}$ and let $\mathcal{U} := \bigcup\{U_n :$ U_n is ctbl$\}$. Then $\mathcal{U}$ is ctbl open being a ctbl union of ctbl open sets. Let $P := X \setminus \mathcal{U}$, so P is closed. It remains to show that P is perfect. Let $V \subseteq P$ be relatively open and nonempty. Then $\exists$ open $V' \subseteq X$ in X s.t. $V = V' \cap P =$ $= V' \setminus \mathcal{U}$. If V contained only one element, V' would be ctbl, so we would have removed it (why? because it's a union of ctbl basic open sets). Thus, V must contain uncthly-many elements. $\square$



HW Discuss Cantor-Bendixson rank and build a closed subspace of $2^{\mathbb{N}}$ of rank $n \in \mathbb{N}$.

HW Well-ordered strictly increasing open sets.

Cor. All closed subsets of any perfect Polish space (e.g. $\mathbb{R}$) satisfy the PSP.

Recall from the optional homework question that every

G_δ subset of a Polish metric space is also Polish up to switching to an equivalent metric. Thus, G_δ subsets also have the PSP.
With a bit of Descriptive Set Theory, we can show that all Borel subsets have the PSP. It can be shown that continuous images of Borel sets also have the PSP, however, whether or not their complements have the PSP is independent of ZFC.

Topological Measure (= Baire Category)

Nowhere dense sets (Review) Let (X, d) be metric space (although all notions below make sense in a topological space).
A set $S \subseteq X$ is call *somewhere dense* if it is dense in some nonempty open set $U \subseteq X$, i.e. $S \cap U$ is dense in U. Otherwise, we say that S is *nowhere dense* (n.d.).

Prop. For a set $S \subseteq X$, TFAE:
(1) S is n.d.
(2) $\forall$ nonempty open U, $\exists$ nonempty open $V \subseteq U$ s.t. $S \cap V = \emptyset$.

(3) $\bar{S}$ is n.d.

(4) $\bar{S}$ has empty interior.

Proof. (1) $\Leftrightarrow$ (2). This is just unraveling the definition of n.d.

(3) $\Rightarrow$ (1). Trivial.

(2) $\Rightarrow$ (3). $\forall$ open $U \neq \emptyset$ $\exists$ open $V \neq \emptyset$ s.t. $S \cap V = \emptyset$, but then $\bar{S} \cap V = \emptyset$ by the def. of closure.

(3) $\Rightarrow$ (4). This is also trivial since a set is dense in its interior.

(4) $\Rightarrow$ (3). Suppose $\bar{S}$ is dense in some open set U. Then $U \subseteq \bar{S}$ hence $U \subseteq \mathrm{int}(S) = \emptyset$, so $U = \emptyset$.

$\square$

Cor. (a) A closed set is n.d. $\Leftrightarrow$ it has $\emptyset$ interior.

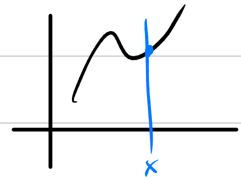
(b) Upgrade: a set is n.d. $\Leftrightarrow$ it's contained in a closed set of $\emptyset$ interior.

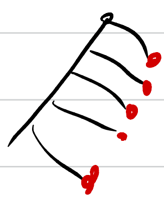
Prop. N.d. sets form an ideal, i.e. they are closed under subsets and finite unions.

Proof. For finite unions, we just need to prove that a union of two n.d. sets is n.d. (because of induction). $\boxed{\text{HW}}$

Examples.

- A singleton $\{x\}$ is a perfect X is n.d. Thus also, any finite set in a perfect X is n.d.
- The graph of any continuous function $f: (0,1) \to \mathbb{R}$ inside $\mathbb{R}^2$. This is because such graphs are closed in $(0,1) \times \mathbb{R}$ (HW problem) and they have empty interior (the x-fibers are singletons).



- $X :=$  $= \{0^n 1^\infty : n \in \mathbb{N}\}$.
  X is n.d. because $\overline{X} = \{0^\infty\} \cup \{0^n 1^\infty : n \in \mathbb{N}\}$ which has empty interior.
- In $\mathbb{N}^{\mathbb{N}}$, the set $X := n^{\mathbb{N}}$ for a fixed $n \in \mathbb{N}$. This is closed (not being in it is witnessed by one coordinate) and has $\emptyset$ interior (open cylinders contain unbdd sequences).

D-tour: examples nonseparable metric spaces: ○ The space $B(X, Y)$ of bdd functions, for X infinite and $|Y| \geq 2$, with the unif. met. In particular, $\ell_\infty(\mathbb{N}) := B(\mathbb{N}, \mathbb{R}) =$ the space of bdd seque.

$\forall\ (x_n), (y_n) \in \ell_\infty(\mathbb{N})$, $\quad d_u((x_n), (y_n)) = \sup_{n\in\mathbb{N}} |x_n - y_n|$.

$\forall\ (x_n), (y_n) \in 2^{\mathbb{N}} \subseteq \ell_\infty(\mathbb{N})$,

$d_u((x_n), (y_n)) = 1 \iff (x_n) \neq (y_n)$, so we have a continuum subset of $\ell_\infty(\mathbb{N})$ of pairwise distance one elements.

Meagre sets. We saw that n.d. sets are closed under finite unions, but clearly not under ctbl unions:

Example. $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} \{q\}$ is everywhere dense in $\mathbb{R}$, while each singleton is n.d.

Def. A subset of a metric space is called **meagre** ([illegible]) if it's a ctbl union of n.d. sets.

Examples.

- $\mathbb{Q}$ in $\mathbb{R}$ is meagre.
- $Q := \{w0^\infty : w \in \mathbb{N}^{<\mathbb{N}}\}$
- $X := \bigcup_{n\in\mathbb{N}} n^{\mathbb{N}}$ is meagre in $\mathbb{N}^{\mathbb{N}}$.

Q

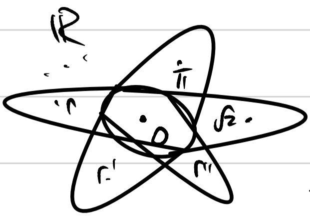
X is the set of all bounded $\mathbb{N}$-valued sequences.

- In $C([0,1], \mathbb{R})$ with the uniform metric, the set of all somewhere differentiable functions (differentiable at least at one point) is meagre.

Warning. A set $S \subseteq X$ can be meagre in X but nonmeagre in a subspace $Y \subseteq X$, i.e. $S \cap Y$ is nonmeagre in Y. E.g. $\mathbb{R} \subseteq \mathbb{R}^2$ is meagre in $\mathbb{R}^2$ but $\mathbb{R} \subseteq \mathbb{R}$ is nonmeagre (as will be implied by Baire Category Theorem). Also, $\{0\}$ in $\mathbb{R}$ is n.d. but $\{0\}$ in $\mathbb{Z}$ is open and nonmeagre.

Example of a separable but not 2^{nd} ctbl topological space.

$X := \mathbb{R}$ but with the following topology:



The basis of this top is the set $\{0\}$ and sets of the form $\{0, r\}$ for each $r \in \mathbb{R} \setminus \{0\}$. Then $\{0\}$ is dense, so X is separable, but it's not 2^{nd} ctbl bc each set $\{0, r\}$ would need to be in every basis.